Shortcutting the diameter of a polygon

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We study the problem of minimizing the diameter of an x-monotone polygon polygon in the plane by attaching a segment to the polygon. Suppose that we attach (or augment) a segment σ to an x-monotone polygon P with n vertices such that both endpoints of σ lie on the boundary of P and the interior of σ does not intersect P. It is considered an addition to the polygon. Then σ (or part of it) can be used as a shortcut to geodesic paths connecting pairs of points in $P \cup \sigma$, and thus the diameter of $P \cup \sigma$ is at most the diameter of P. We call such a segment a *shortcut* for P. Thus, our problem is to find a shortcut σ for P minimizing the diameter of $P \cup \sigma$. We present an $O(n)$-time algorithm to determine whether the diameter of an x-monotone polygon with n vertices can be reduced by attaching a horizontal segment to the polygon under the L_1 metric. We also present an $O(n \log n)$-time algorithm using $O(n)$ space for finding a shortcut that minimizes the diameter.

First, we present an $O(n)$-time algorithm to determine whether P has a useful shortcut. We say that a shortcut σ is *useful* if $\mathsf{diam}(P \cup \sigma) < \mathsf{diam}(P)$. We call a vertex v an *anchor* of P if, for any diametral pair (α, β) of P, every geodesic path connecting α and β passes through v. We show that P has a useful shortcut if and only if P has an anchor. Then we show how to find the anchors of P in $O(n)$ time.

Next, we present an algorithm that computes an optimal shortcut for P. We say a shortcut σ is *valid* if σ contains an anchor of P in the range of its x-coordinates. Let S denote the union of all valid shortcuts. We partition S into trapezoids using horizontal segments, and find the trapezoids that contain optimal shortcuts using a binary-search-like algorithm. For the binary-search-like algorithm, we define two functions f and g for shortcuts such that for a fixed shortcut σ, the diameter of $P \cup \sigma$ is $\max\{f(\sigma), g(\sigma)\}$. Here, we can compute $f(\sigma)$ in $O(n)$ time in the worst case, and $g(\sigma)$ in $O(\log n)$ time with help of appropriate query data structures. We show that $f(\sigma)$ satisfies a certain monotonicity such that $f(\sigma) \leq f(\sigma')$ holds for shortcuts σ and σ' satisfying a certain ordering; the projection of σ onto x axis is strictly contained in the projection of σ'. We also show a certain unimodality of g.

To find the trapezoids that contain optimal shortcuts efficiently, we use $O(n)$ segments of S that is incident to a vertex of the partition. We call such a segment a *critical* shortcut. We compute $g(\sigma)$ for every critical shortcut σ and the local minimum of g in each trapezoid in $O(n \log n)$ time. Then we compute $f(\sigma)$ for critical shortcuts σ with a binary-search-like algorithm to find those trapezoids containing optimal shortcuts. This can be done in $O(n \log n)$ time in total. Then, for each such trapezoid, we find a shortcut minimizing the diameter. Again, using the monotonicity of f and the unimodality of g, we apply binary search in each trapezoid and find such a shortcut. In total, this takes $O(n \log n)$ time and $O(n)$ space. Finally, we return the one achieving the minimum diameter.

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