# Protecting the Connectivity of a Graph Under Non-Uniform Edge Failures

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We study the problem of guaranteeing the connectivity of a given graph by protecting or strengthening edges. Herein, a protected edge is assumed to be robust and will not fail, which features a non-uniform failure model. We introduce the $(p, q)$-Steiner-Connectivity Preservation problem ($(p, q)$-SCP) where we protect a minimum-cost set of edges such that the underlying graph maintains $p$-edge-connectivity between given terminal pairs against edge failures, assuming at most $q$ unprotected edges can fail. We also consider special cases $(p, q)$-$s$-$t$-Connectivity Preservation ($(p, q)$-stCP) where there is one terminal-pair, and $(p, q)$-Global-Connectivity Preservation ($(p, q)$-GCP) where each pair of vertices are terminal-pairs.

We design polynomial-time exact algorithms for the cases where $p$ and $q$ are small, including $(p, 1)$-SCP, $(1, 2)$-SCP and $(2, 2)$-GCP. The first result for $(p, 1)$-SCP is obtained by observing that a minimal solution contains all edges in any terminal-separating $p$-edge-cut. The polynomial-time algorithm for $(1, 2)$-SCP relies on a decomposition of terminal-separating cuts of size 2, based on which we devise a divide-and-conquer algorithm. For $(2, 2)$-GCP, we show a reduction to a multi-commodity flow problem on a tree, using the well-known tree-representation of minimum cuts [Dinits et al., 1976]. Then we solve the multi-commodity flow problem by dynamic programming.

When both $p$ and $q$ are part of the input, we prove that even deciding whether a given solution is feasible is NP-complete, which implies there is no polynomial-time $\alpha$-approximation for any $\alpha > 0$ unless $\mathsf{P} \neq \mathsf{NP}$. This hardness also carries over to Flexible Network Design, a research direction that has gained significant attention. In particular, previous work focuses on problem settings where either $p$ or $q$ is constant [Chekuri et al., 2023], for which our new hardness result now provides justification.

On the positive side, we show an $O(q \cdot \log p)$-approximation for $(p, q)$-SCP based on a primal-dual framework [Williamson et al., 1995], assuming $p$ is constant. We improve the approximation ratio to $O(\log p \min\{\log n, p + q\})$ for $(p, q)$-GCP, without any restriction on $p$ or $q$.

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