Dynamic Maximum Depth of Geometric Objects

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Given a set $\mathcal{O}$ of geometric objects in the plane (rectangles, squares, disks etc.), the *maximum depth* (or *geometric clique*) of $\mathcal{O}$ is the largest depth of any point in the plane, where the depth of a point p, denoted $\mathsf{dep}(p, \mathcal{O})$, is defined as the number of objects in $\mathcal{O}$ containing p. The maximum depth of the set $\mathcal{O}$ is written as $\mathsf{dep}(\mathcal{O}) = \max_{p \in \mathbb{R}^d} \mathsf{dep}(p, \mathcal{O})$. In this paper, we study the *dynamic* maximum-depth problem for (axis-parallel) rectangles and disks, designing exact and approximate dynamic maximum-depth data structures with sublinear update time. We achieve the following results:

Theorem 1. *For every $k \in \mathbb{Z}^+$, there exists a $\frac{1}{k}$-approximate dynamic maximum-depth data structure for axis-parallel rectangles with $O(n^{1/(k+1)} \log n)$ amortized update time.*

When $k = 1$, the above theorem gives us an exact data structure for dynamic maximum depth of rectangles, with $O(\sqrt{n} \log n)$ update time. Up to logarithmic factors, this bound matches the update time of the best known *offline* dynamic data structure for the problem, which is $O((\sqrt{n}/\sqrt{\log n}) \cdot \log^{3/2} \log n)$. Furthermore, whether the static maximum-depth problem for rectangles can be solved in $O(n^{1.5-\varepsilon})$ time is a longstanding open problem. Thus, $\widetilde{O}(\sqrt{n})$ update time is the best one can expect at this point, even in the insertion-only version.

Theorem 2. *For any constant $\varepsilon > 0$, there exists a $(\frac{1}{2} - \varepsilon)$-approximate dynamic maximum-depth data structure for disks with $n^{2/3} \log^{O(1)} n$ amortized update time.*

The key idea underlying this data structure is to reduce the original problem to dynamic *discrete* maximum depth (for disks), by introducing a $\frac{1}{2} - \alpha$ multiplicative error. In the discrete version of the problem, besides a dynamic set $\mathcal{D}$ of disks, we also have a dynamic set S of points in $\mathbb{R}^2$, and what we want to maintain is $\max_{p \in S} \mathsf{dep}(p, \mathcal{D})$. This kind of reduction works for not only disks, but also general fat objects (where the multiplicative error depends on the fatness of the objects). We believe that this idea is of independent interest and may find applications in other problems as well. Moreover, it worth noting that the static maximum-depth problem for disks is known to be 3SUM-hard and thus has the conjectured lower bound $\widetilde{\Omega}(n^2)$. As such, it is unlikely to have an exact dynamic maximum-depth data structure for disks with (truly) sublinear update time, even in the insertion-only setting.

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