Near-unit distance embedding of points in the plane and space

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The problem of embedding points satisfying a geometric constraint is a basic problem of geometry. It relates to the information visualization. For example, the force-directed method of graph drawing (e.g., Tutte's embedding algorithm [2] and Sugiyama's algorithm [3]) embeds vertices of a graph into the Euclidean plane, and regards each edge of a graph as a spring to control the Euclidean distance between the points approximating desired edge lengths of the graph, and considers optimization problems.

Let us consider the embedding problem of a complete graph K_n so that each edge has the same (say, unit) distance. In other words, if vertices are embedded to the points $p_1, p_2, \ldots, p_n$, the Euclidean distance $d_{i,j}$ between p_i and p_j is exactly 1 for each ordered pair $i < j$. Unfortunately, we can only embed K_2 and K_3 in such a perfect fashion in the plane.

Now, consider an objective function for a set of n points $P = \{p_1, p_2, \ldots, p_n\}$

$$f_{p,q}(P) = f_{p,q}(p_1, p_2, \ldots p_n) = \sum_{i<j} |d^p_{i,j} - 1|^q$$

for some predetermined positive real numbers p and q. $f_{p,q}(P) = 0$ for a perfect embedding P with all unit distances, and we consider the optimal embedding P to minimize $f_{p,q}(P)$ as its approximation. We allow the concentration of points, that is, more than one point can be located at the same position. Asano [1] considered this problem of embedding to the one-dimensional space.

In this paper, we investigate the two and three dimensional embedding to minimize $f_{2,2}(P)$.

Theorem 1. *For the two dimensional case, let P be a configurtion of n points to minimize $f_{2,2}(P)$. Then either*

1. *P is a set of points concentrated on at most 5 points, or*
2. *P is a configuration of points on a circle with radius $\frac{1}{\sqrt{3}}$.*

We further conjecture that $\frac{n^2}{6} - \frac{n}{2}$ is indeed the minimum if $n \geq 3$. We can observe that $f_{2,2}(P) = \frac{n^2}{6} - \frac{n}{2}$ is attained by any configuration P on $C(\sqrt{1/3})$ satisfying a condition on the covariance matrix of coordinate values of the points. This implies that there are various kinds of solutions if the conjecture is true.

For example, if n is a multiple of 3, both regular n-gon and a concentrated 3 point configuration represented by an equilateral triangle on the circle attains the same optimal value of the objective function. This is counter-intuitive. Indeed, if we compare these two configurations with respect to the modified objective function $f_{2,q}$, the regular n-gon is better if $q > 2$, while the triangle is if $q < 2$. Hence, a phase transition is observed at the value $q = 2$.

The following holds for the three dimensional case:

Theorem 2. *The optimal location of the set P of $n \geq 4$ points must satisfy one of the followings if we allow fractional multiplicities.*

1. *A set of at most 7 points with multiplicities,*
2. *A set on a sphere of radius $\sqrt{3/8}$, or*
3. *Union of a set on a circle C and at most three multiple points on the line perpendicular to the plane containing C and going through the center of C.*

Moreover, if the point set lies on the sphere of radius $\sqrt{3/8}$, $f_{2,2}(P) = n^2/8 - n/2$.

We conjecture that $n^2/8 - n/2$ is indeed the minimum. The value $n^2/8 - n/2$ can be attained without fractional multiplicities if n is even and $n \geq 4$.

We also give some experimental results to support our conjectures.

References

[1] T. Asano, Arranging points on the line so that pairwise distances are close to 1, *working paper.*

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